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A Direct Energy Balance Method for Approximating Envelope Decay of Oscillating MEMS Structures

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***Abstract*—The real pole component (envelope decay coefficient) of an oscillating microelectromechanical systems (MEMS) structure is calculated directly in the energy domain without using an equation of motion. Similar to the simplified Rayleigh frequency calculation in which maximum potential and kinetic energy are equated, our method equates the initial minus dissipated energy to present energy.**

I. Introduction

When designing MEMS integrated circuits (ICs), it is desirable to find transfer functions to model the motional structures such that they are easily combined with models of the electronic activation and sensing circuits. For motional structures, a popular way of approximating the resonance frequency is the simplified Rayleigh method [1]. With this approach, it is unnecessary to construct a differential equation of motion; rather the simpler idea of calculating potential and kinetic energy of the structure under the constraint of energy conservation is used. Although the Rayleigh method provides the imaginary components of the poles of the motional structure, it is desirable to find a comparably simple energy balancing method as a counterpart to approximate the real component of the poles that characterize the damped envelope decay of free oscillation. Presently, the real component is typically found by indirectly forming an equation of motion by using the more complex Rayleigh dissipation function in the Lagrange equation [2], or by fitting an energy equivalent viscous damping coefficient into an equation of motion [1], or by invoking the energy relation of the definition of Q [3]. However, we show that the original basic energy balancing idea of the simplified Rayleigh resonant frequency calculation can be extended by developing an alternative approximation of envelope decay. The derivation for our method is performed on a flexible cantilever or beam with arbitrary anchors as defined by a deflection equation using, e.g., polynomial coefficients. As will be apparent after the derivation, the method can be adapted easily to analyze other structures.

II. Background

We start with a brief overview of the simplified Rayleigh frequency method as applied to a flexible cantilever or beam with arbitrary anchors. Assume that the dynamic structure deformity $\tilde{y}$ is identical to the static deformation y, which is normalized to the maximum deflection A:

$$\tilde{y} = yA\sin(\omega t) \quad \text{where } y \equiv y(x). \tag{1}$$

When the total system energy is kinetic and none is stored potentially we have:

$$\mathrm{KE_{MAX}} = \int_0^L \frac{m}{2}\left(\frac{\partial \tilde{y}}{\partial t}\right)^2 dx\Big|_{\mathrm{MAXIMUM\_VELOCITY}}, \tag{2}$$

which occurs when the structure crosses the zero deflection point, where m is mass per unit length and L is the length of the structure. When all the energy is stored with the accompanying pause in motion, we have for the potential energy:

$$\mathrm{PE_{MAX}} = \int_0^L \frac{EI}{2}\left(\frac{\partial^2 \tilde{y}}{\partial x^2}\right)^2 dx\Big|_{\mathrm{MAXIMUM\_DEFLECTION}}, \tag{3}$$

where E is Young's modulus and I is the relevant moment of inertia.

Because the energy of a nondamped system is conserved, we can equate (2) and (3), which after some calculation yields:

$$\omega_0 = \sqrt{\frac{EI\int_0^L \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx}{m\int_0^L y^2 dx}} \tag{4}$$

for resonant frequency.

III. Energy Balance Envelope Decay Method

Assume now that the system described above is mildly damped such that the resonant frequency is negligibly affected. Eq. (1) will need to be modified so that envelope amplitude A is now time varying yielding,

$$\tilde{y} = yA(t)\sin(\omega t). \tag{5}$$

The total system energy also is no longer constant, but can be written as:

$$E_T(t) = E_i - \iint_{x\,y} k_d \frac{\partial \tilde{y}}{\partial t} \bullet dy dx = E_i - k_d \int_x\int_0^t \left(\frac{\partial \tilde{y}}{\partial \tau}\right)^2 d\tau dx, \tag{6}$$

where E_i is the initial energy in the system and the integral terms represent the energy dissipated by the system

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through viscous damping along the length of the structure. k_d is the damping coefficient per unit length and can include effects such as squeeze film [4], [5] or Stokes-type fluid shear damping [3]. The total system energy also can be written at the time points of maximum deflection, which will correspond naturally to points along the envelope $A(t)$. Replacing $\tilde{y}$ by using (5) instead of (1) in (3), and simplifying yields:

$$E_T(t)\Big|_{t=\text{Integer\_Cycle\_Interval}} = \frac{1}{2}EI(A(t))^2 \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx. \tag{7}$$

If we note that the integral in (7) is constant, $E_T \equiv E_T(A(t))$ at the integer cycle sample points of maximum deflection. Notice, however, that between the maximum deflection time points that exactly define $A(t)$, (7) is monotonic. Thus, the constraint that (7) is sampled can be removed and $E_T \equiv E_T(A(t))$ approximates the envelope at all times.

Returning to (6), we substitute from (5) and simplify to yield:

$$E_T(t) = E_i - k_d \int_0^L y^2 dx \left[\int_0^t (A(\tau))^2 \omega^2 \cos^2(\omega\tau) d\tau + \int_0^t \left(\frac{dA(\tau)}{d\tau}\right)^2 \sin^2(\omega\tau) d\tau \right]. \tag{8}$$

For a mildly damped system, $A(\tau)$ varies slowly with respect to the squared trigonometric functions. So, to refine (8), the mean squared values for these functions can be substituted to obtain (see Appendix for more formal treatment) the quasistatic envelope approximation:

$$E_T(t) \approx E_i - \frac{k_d}{2} \int_0^L y^2 dx \int_0^t \left(\omega^2 (A(\tau))^2 + \left(\frac{dA(\tau)}{d\tau}\right)^2\right) d\tau. \tag{9}$$

Put $EI \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \to K_e$ in (7) and $k_d \int_0^L y^2 dx \to K_d$ in (9). Then equating a nonsampled version of (7) with (9) and differentiating, we may write:

$$\frac{dA(t)}{dt} = -\frac{K_d \omega^2}{2K_e} A(t) - \frac{K_d}{2K_e} \frac{1}{A(t)} \left(\frac{dA(t)}{dt}\right)^2. \tag{10}$$

This equation may be solved by recognizing the derivative as the quadratic variable to yield:

$$\frac{dA(t)}{dt} = \left[-\frac{K_e}{K_d} \pm \sqrt{\frac{K_e^2}{K_d^2} - \omega^2} \right] A(t). \tag{11}$$

Therefore, the solution is simply:

$$A(t) = Ce^{\sigma t}, \quad \text{where } \sigma = -\frac{K_e}{K_d} + \sqrt{\frac{K_e^2}{K_d^2} - \omega_0^2}, \tag{12}$$

and the positive square root is retained since σ must approach zero when K_d, the variable containing damping coefficient k_d, also approaches zero. To be consistent with (4), ω_0 replaces ω, indicating we are interested in resonance when finding the poles of the structure's transfer function $p_{1,2} = \sigma \pm j\omega_0$. By inspecting (4), (12) also can be written as:

$$\sigma = -\frac{m}{k_d}\omega_0^2 + \sqrt{\left(\frac{m}{k_d}\right)^2 \omega_0^4 - \omega_0^2}. \tag{13}$$

We now show how our method can be adapted to other structures. Reviewing (7) and (9), notice that K_e and K_d as defined capture the relevant functions of normalized static deflection of the linear structure. The remaining parts of (7) and (9) perform functions of time varying scaling of K_e and K_d. For an arbitrary structure to be used in our formulation, all components of the structure would have functions subjected to the same scaling. Knowing energy is additive, we can revise K_e and K_d to capture normalized static deflection functions of a structure consisting of a group of both flexible beams and rigid bodies, and apply the new coefficients in (12) to find σ:

$$K_e = \sum_{i=1}^{n} EI \int_0^{L_i} \left(\frac{d^2y_i}{dx^2}\right)^2 dx, \tag{14}$$

$$K_d = \sum_{i=1}^{n} k_d \int_0^{L_i} y_i^2 dx + \sum_{i=1}^{m} c_i\, \hat{y}_i^2, \tag{15}$$

As in the derivation of the single beam or cantilever, $y_i \equiv y_i(x)$, but $\hat{y}_i$ is not a function because it represents the displacement of a rigid body within the composite structure. Note that all y_i and $\hat{y}_i$ need to be normalized to the maximum deflection of the composite structure as bounded by $A(t)$. The set of c_i are lumped parameter damping coefficients for the rigid bodies. Naturally, rigid bodies need not be accounted for within K_e because they do not store potential energy. When it is desirable to incorporate flexible beams with nonuniform properties along the length (i.e., varying thickness), constants E, I, and k_d can be transformed to functions of x and placed in the integrals of (14) and (15).

IV. Results

Our energy balancing method, along with the energy equivalent viscous damping coefficient method (EEVD) [1], are used to calculate σ. This is done as a validity check as well as to form a comparison. A 2 μm × 2 μm × 800 μm polysilicon beam situated between two rigid beams, each having a 1-μm gap, are used in the calculations. The rigid beams might serve as capacitor plates that sense the center beam's position [if an alternating current (AC) were forced through the center beam, for example, a B-field could be measured [6]]. The normalized static deflection is $y = 16x^2/L^2 - 32x^3/L^3 + 16x^4/L^4$ (assumed to be the

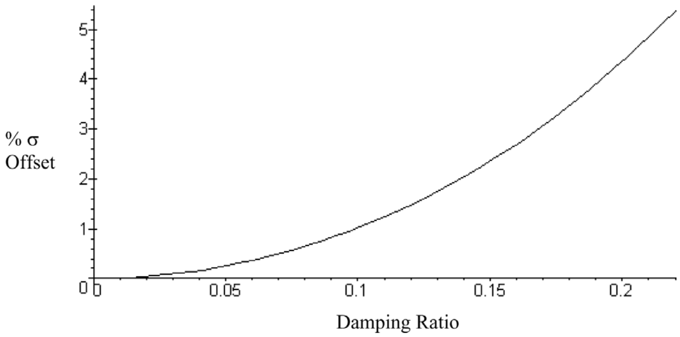

Fig. 1. Percentage offset of our energy balance σ relative to the EEVD σ versus the standard damping ratio. The physical beam parameters are implicit variables.

same shape as in oscillation). Also, $EI = .2133N\ \mu\mathrm{m}^2$, $m = 9.2$ pg/μm, and $k_d = 592e - 18N$ s/$\mu\mathrm{m}^2$, as caused by squeeze film damping with the rigid beams. Using Rayleigh's method, $\omega_0 = 169$ krad/s. With our energy balance approximation, $\sigma = -33432\ \mathrm{s}^{-1}$, and with EEVD, $\sigma = -32174\ \mathrm{s}^{-1}$. We note that EEVD, which maps the continuous physical parameters of the beam onto a second order lumped parameter differential equation, yields: $p_{1,2} = \sigma_{EEVD} \pm j\sqrt{\omega_0^2 - \sigma_{EEVD}^2} = -(k_d/(2m)) \pm j\sqrt{\omega_0^2 - (k_d/(2m))^2}$, where ω_0 is the same as in (4). Conversely, the couplet of Rayleigh's simplified frequency calculation and our method yields a free standing $\pm j\omega_0$ for the imaginary pole component and the real component σ, (13), contains the radical. Thus, the solutions of the two methods have comparable complexity with a slight shift in numerical value under mild damping conditions. We can write the damping ratio in terms of σ_{EEVD}, $\varsigma = -\sigma_{EEVD}/\omega_0$. By inspection we then can write (13) as a function of ς, $\sigma = (\omega_0/2)\left((-1/\varsigma) + \sqrt{(1/\varsigma^2) - 4}\right)$. We then plot the percentage offset between the two methods relative to EEVD versus ς, $100\,(\sigma - \sigma_{EEVD})\,/\sigma_{EEVD}$, depicted in Fig. 1. Note that ω_0, though part of both our method and EEVD calculations, cancels out of the computation of Fig. 1. The mild damping ratio range plotted easily exceeds what would be considered reasonable for a structure operating as a resonator, yet the maximum offset is bounded by $\approx 5\%$, thus further demonstrating the numerical accuracy of our σ approximation method.

V. Conclusions

Not only the imaginary, but, as was demonstrated, the real pole component of an oscillating MEMS structure can be calculated directly in the energy domain, thus bypassing the construction of a differential equation of motion.

Appendix A

We give an overview of the quasistatic envelope approximation used to arrive at (9). For a function $f(t)$ that varies slowly with respect to $\cos(\omega t)$, as in a mildly damped system, we can write:

$$\begin{aligned}\int_0^{t=\frac{2\pi}{\omega}n} f(\tau)\cos^2(\omega\tau)d\tau &\approx \sum_{i=0}^{n} f\left(\frac{2\pi}{\omega}i\right)\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega\tau)d\tau \\ &= \frac{1}{2}\sum_{i=0}^{n} f\left(\frac{2\pi}{\omega}i\right)\frac{2\pi}{\omega} \\ &\approx \frac{1}{2}\int_0^{t=\frac{2\pi}{\omega}n} f(\tau)d\tau.\end{aligned}$$

The above applies if $\sin(^\circ)$ is substituted for $\cos(^\circ)$. Also, since $f(t)$ varies slowly, t in the first and last integrals need not be constrained to $t \equiv t(n)$.

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